

ON THE CLASSIFICATION OF GALOIS OBJECTS FOR FINITE GROUPS

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ABSTRACT. We classify Galois objects for the dual of a group algebra of a finite group over an arbitrary field.

1. INTRODUCTION

Let H be a Hopf algebra. A right H -comodule algebra A is called a right H -Galois object if the map $\text{can}: A \otimes A \rightarrow A \otimes H, a \otimes b \mapsto ab_{(0)} \otimes b_{(1)}$ is bijective. When the Hopf algebra H is the dual of a group algebra of a finite group, then Galois objects are the G -Galois extensions of non-commutative rings introduced by Chase, Harrison and Rosenberg [3].

The Galois objects are very interesting because have a categorial interpretation, and they can be used for the construction of new Hopf algebras. Let ${}^H\mathcal{M}$ be the category of finite dimensional left H -comodules. A fiber functor $F : {}^H\mathcal{M} \rightarrow \text{Vec}_k$ is an exact and faithful monoidal functor that commutes with arbitrary colimits. Ulbrich defined in [14] a fiber functor F_A associated with each H -Galois object A , in the form $F_A(V) = A \square_H V$, where $A \square_H V$ is the cotensor product over H of the right H -comodule A and the left H -comodule V . In *loc. cit.* was defined a bijective correspondence between isomorphism classes of H -Galois objects and isomorphism classes of fiber functors of ${}^H\mathcal{M}$.

Given a Hopf algebra H and a left H -Galois object A , there is a new Hopf algebra associated $L(A, H)$, see [11]. The Hopf algebra $L(A, H)$ is the Tannakian-Krein reconstruction from the fiber functor associated to A , [11, Theorem 5.5].

It is well known that the classification of H^* -Galois objects for a finite dimensional Hopf algebra H , is equivalent to the classification of twist in H , see [1, Section 4]. In [10], Movshev classified the twist for complex group algebras, so this implies the classification of Galois objects for the dual of a complex group algebra of a finite group.

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Throughout this article we work over an arbitrary field k , and we will denote by k^G the dual of the group algebra for a finite group G .

The aim of this paper is to classify k^G -Galois objects, generalizing the classification of Movshev [10], and Davydov [4].

To formulate our main result we first describe a Galois datum associated to a finite group G and a field k .

Definition 1.1. A *Galois datum associated to k^G* is a collection $(S, K, N, \sigma, \gamma)$ such that

- i) S is a subgroup of G and N is a normal subgroup of S .
- ii) $K \supseteq k$ is a Galois extension with Galois group S/N .
- iii) $\text{char}(k) \nmid |N|$.
- iv) $\sigma : N \times N \rightarrow K^*$ is a non-degenerate 2-cocycle.
- v) $\gamma : S \times N \rightarrow K^*$ satisfies the equations (C1), (C2), and (C3).

A 2-cocycle $\sigma \in Z^2(N, K^*)$ is called non-degenerate if $\sigma(s, t) = \sigma(t, s)$ for all $t \in C_N(s)$ implies $s = 1$. Equivalently, σ is non-degenerate if the center of the twisted group algebra $K_\sigma N$ coincides with K . Also, by Theorem of Maschke for twisted group algebras [7, Theorem 2.10, pag 85], if $\text{char}(k) \nmid |N|$ and σ is non-degenerate then $K_\sigma N$ is a central simple algebra over K .

The function γ is univocally determined by a Hochschild 1-cocycle $\gamma \in HZ^1(S, C^1(N, K^*))$, where $C^1(N, K^*)$ is an S -bimodule with actions given by the equations (10), and (9), see Section 7. In the Section 7 we define some cohomological obstructions which establish necessary conditions for the existence of γ .

Let $(S, K, N, \sigma, \gamma)$ be a Galois datum associated to k^G . The group S acts over K through the canonical projection to $\text{Gal}(K|k) = S/N$. We will denote by $A(K_\sigma N, \gamma)$ the twisted group algebra $K_\sigma N$ with S -action defined by

$$g \rightharpoonup \alpha u_x = \bar{g}(\alpha) \gamma(g, x) u_{gx},$$

for $g \in S$, $x \in N$, and $\alpha \in K$.

In the Subsection 3.1, we study the induction functor from the category of S -algebra to the category of G -algebras. Using the induction functor we define the G -algebra $\text{Ind}_S^G(A(K_\sigma N, \gamma))$.

Now we can formulate our main result.

Theorem 1.2. *Let G be a finite group and let k be a field.*

- i) *Let $(S, K, N, \sigma, \gamma)$ be a Galois datum associated to k^G . Then the G -algebra $\text{Ind}_S^G(A(K_\sigma N, \gamma))$ is a k^G -Galois object.*
- ii) *Let A be a k^G -Galois object. Then $A \simeq \text{Ind}_S^G(A(K_\sigma N, \gamma))$ for a Galois datum $(S, K, N, \sigma, \gamma)$.*

In the Subsection 5.1 we establish an equivalence among the Galois data, this equivalent data define isomorphic Galois objects.

The paper is organized as follows: in Section 2 we give a brief account of results on Hopf-Galois extensions that will be needed in the sequel. In Section 3 we show that every k^G -Galois object is the induced of a simple Galois object. For the proof we define the concept imprimitive system for G -algebras. In Section 4 we show that every simple k^G -Galois object is a twisted group algebra. For this proof we use as mean tool the Miyashita-Ulbrich action. In Section 5 we define k^G -Galois objects associates to some group-theoretical data. In Section 6 we proof our main result, see Theorem 1.2. Also, we give some non-trivial examples of Galois objects. In Section 7 we present some cohomological obstruction for the existence of a function γ , see Definition 1.1.

2. PRELIMINARIES ON HOPF GALOIS EXTENSIONS

In this section we review some results on Hopf Galois extensions that we will need later. We refer the reader to [12] for a detailed exposition on the subject.

Definition 2.1. Let H be a Hopf algebra. Let also A be a right H -comodule algebra with structure map $\rho : A \rightarrow A \otimes H$, $\rho(a) = a_{(0)} \otimes a_{(1)}$, and let $B = A^{\text{co}H}$. The extension $A \supseteq B$ is called a right *Hopf Galois* extension, or a right *H-Galois* extension if the canonical map

$$\text{can} : A \otimes_B A \rightarrow A \otimes H, \quad a \otimes b \mapsto ab_{(0)} \otimes b_{(1)},$$

is bijective.

A right H -Galois extension of the base field k will be called a right *H-Galois object*. Left H -Galois extensions and left H -Galois objects are defined similarly.

Example 2.2. Let the Hopf algebra H coact on itself through the comultiplication Δ . Then the canonical map $H \otimes H \rightarrow H \otimes H$, $x \otimes y \mapsto xy_{(1)} \otimes y_{(2)}$ is bijective with inverse $x \otimes y \mapsto xS(y_{(1)}) \otimes y_{(2)}$.

Assume from now on that H is finite dimensional Hopf algebra.

We have a characterization of H^* -Galois objects A in terms of the natural H -action over A :

Proposition 2.3. *A is a right H^* -Galois object if and only if A is finite dimensional and the map $\theta : A \# H \rightarrow \text{End } A$, $\theta(a \# h)(b) = a(h \cdot b)$ is an isomorphism.*

Proof. See [9, 8.3.3 Theorem]. □

Proposition 2.4. *Every right H -Galois object is isomorphic to a crossed product $H^\sigma = k \#_\sigma H$, where $\sigma : H \otimes H \rightarrow k$ is an invertible 2-cocycle. Moreover, if H is semisimple then H^σ is semisimple.*

On the other hand, H^σ is a right H -Galois object for all such σ .

Proof. Let A be a right H -Galois object. Since $A \# H^* \simeq \text{End } A$ is simple artinian, thus by [9, Proposition 8.3.6] there is $\sigma : H \otimes H \rightarrow k$ an invertible 2-cocycle such that

$$A \simeq k \#_{\sigma} H.$$

Now by [9, Theorem 7.4.2], if H is semisimple and finite dimensional then $A \simeq k \#_{\sigma} H$ is semisimple. \square

Remark 2.5. It follows by the Proposition 2.4, that for an H -Galois object A , $\dim_k(A) = \dim_k H$.

2.1. Miyashita-Ulbrich action. Let $A \supseteq B$ be an H -Galois extension. The Miyashita-Ulbrich action of H on the centralizer A^B of B in A , makes A^B into a commutative algebra in \mathcal{YD}_H^H the category of right Yetter-Drinfeld modules.

Definition 2.6. Let H be a Hopf algebra, and A an H -Galois extension of B . The Miyashita-Ulbrich action of H on A^B is defined by $a \triangleleft h = h^{[1]} a h^{[2]}$, $a \in A^B$, $h \in H$, where $\text{can}(h^{[1]} \otimes h^{[2]}) = 1 \otimes h$.

The Miyashita-Ulbrich action of H on the H -Galois object A is characterized as the unique map $A \otimes H \rightarrow A$, $a \otimes h \mapsto a \triangleleft h$, such that $ab = b_{(0)}(a \triangleleft b_{(1)})$, for all $a, b \in A$.

Example 2.7. Consider the H -Galois object $A = H$ as in Example 2.2. In this case, the Miyashita-Ulbrich action coincides with the right adjoint action of H on itself: $a \triangleleft h = \mathcal{S}(h_{(1)}) a h_{(2)}$, $a, h \in H$.

Therefore a Hopf subalgebra $H' \subseteq H$ is a normal Hopf subalgebra if and only if it is stable under the Miyashita-Ulbrich action.

The next theorem will be useful to reduce the classification of H -Galois object in the case that we have an exact succession $k \rightarrow K \rightarrow H \rightarrow Q \rightarrow k$ of Hopf algebras.

Theorem 2.8. Let H be a finite dimensional Hopf algebra, $K \subseteq H$ a normal Hopf subalgebra, and $Q = H/K^+H$ the quotient Hopf algebra. Let $\delta : A \rightarrow A \otimes H$ be an H -comodule algebra. Then the following are equivalent:

- i) A is an H -Galois object.
- ii) $A^{\text{co}Q}$ is a K -Galois object, and $A^{\text{co}Q} \subseteq A$ is a faithfully flat Q -Galois extension.

Proof. See [12, Theorem 4.5.1] \square

In the case that $H = k^G$ for a finite group G , and A is a simple algebra we have:

Corollary 2.9. Let A be a k^G -comodule algebra, where A is a simple algebra with center K . Then A is a k^G -Galois object if and only if A is a K^N -Galois object (where $N = \{x \in G \mid x \cdot \alpha = \alpha, \forall \alpha \in K\}$) and K is a $k^{G/N}$ -Galois object.

Proof. Suppose that A is a k^G -Galois object. If we show that $A^N = K$, the necessity follows by Theorem 2.8.

Since G acts over A and $K = \mathcal{Z}(A)$, then we have a homomorphism $\pi : G \rightarrow \text{Aut}_k(K)$, such that $\ker \pi = N$. Thus π induces an injective homomorphism $\tilde{\pi} : G/N \rightarrow \text{Aut}_k(K)$. Now, $A^G = k$ implies $K^{G/N} = k$. Hence by [2, Theorem 13] we have $[K : k] \geq |G/N|$.

By Theorem 2.8 A^N is a $k^{G/N}$ -Galois object, so $\dim_k A^N = |G/N|$. Since $K \subseteq A^N$, and $[K : k] \geq |G/N|$, thus $A^N = K$.

The Sufficiency follows by Theorem 2.8. \square

3. GALOIS EXTENSIONS OF FINITE GROUPS

Let G be a finite group, and let A be a G -algebra with G -action $kG \otimes A \rightarrow A$, $g \otimes a \mapsto g \rightharpoonup a$. Equivalently, A is a k^G -comodule algebra with respect to the coaction $\rho : A \rightarrow A \otimes k^G$ defined by

$$\rho(a) = \sum_{g \in G} g \rightharpoonup a \otimes t_g,$$

where $(t_g)_{g \in G}$ denotes the basis of k^G consisting of the canonical idempotents $t_g(h) = \delta_{g,h}$ (Kronecker's delta).

The next proposition is a restatement of [4, Proposition 3.1] when k is not an algebraically closed field.

Proposition 3.1. *Let A be a G -algebra. Then A is a k^G -Galois object if and only if A satisfies the following conditions:*

- i) $\dim(A) = |G|$.
- ii) A has no non-trivial G -invariant left ideals.
- iii) $A^G = k$.

Proof. Let J be a G -invariant left ideal of A . By [12, Theorem 2.3.9] the map $A \otimes (A/J)^G \rightarrow A/J$, $a \otimes m \mapsto a \cdot m$ is an isomorphism. Now, if $J \neq 0$, then $A/J = 0$, so $A = J$.

Conversely, we can consider A as a left $A \rtimes G$ -module with structure map θ . By (ii), A is a simple $A \rtimes G$ -module. Let $D = \text{End}_{A \rtimes G}(A)$. Applying the Jacobson Density Theorem (see [8, F20, pag 139]), we have that natural homomorphism

$$\theta : A \rtimes G \rightarrow \text{End}_D(A) = \text{End}_{A^G}(A) = \text{End}_k(A)$$

is surjective, then by (i) θ is bijective. \square

Remark 3.2. The condition $A^G = k$ in Proposition 3.1 is not necessary if k is algebraically closed.

The following corollary is an alternative proof of the Proposition 2.4 for the case $H = k^G$.

Corollary 3.3. *Every k^G -Galois object is a semisimple algebra.*

Proof. Let A be a k^G -Galois object. Since the Jacobson radical $\text{rad}(A)$ of the algebra A is a left G -invariant ideal, we conclude that $\text{rad}(A) = 0$. \square

3.1. Imprimitive algebras and induced algebras. In this subsection we introduce the notion of imprimitive algebra.

Definition 3.4. Let A be a G -algebra and $\{e_1, \dots, e_n\}$ a set of orthogonal central idempotents in A , such that $1_A = e_1 + \dots + e_n$. We will say that $\{e_1, \dots, e_n\}$ is an *imprimitive system* if $g \rightharpoonup e_i \in \{e_1, \dots, e_n\}$ for all $g \in G, i = 1, \dots, n$.

A G -algebra is called *imprimitive* if there is a non-trivial imprimitive system in A .

Let S be a subgroup of G and (B, \cdot) an S -algebra. The vector space $kG \otimes_{kS} B$ with G -action and multiplication defined by:

$$\begin{aligned} g \rightharpoonup (h \otimes x) &= gh \otimes x \\ (g \otimes x)(h \otimes y) &= \begin{cases} h \otimes (h^{-1}g \cdot x)y & \text{if } h^{-1}g \in S \\ 0 & \text{if } h^{-1}g \notin S, \end{cases} \end{aligned}$$

is a G -algebra for $g, h \in G$ and $x, y \in B$

Also, consider the algebra of functions

$$A_S(G, B) = \{r : G \rightarrow B \mid r(sg) = s \cdot r(g) \quad \forall s \in S, g \in G\},$$

which is a G -algebra with action defined by $(g \rightharpoonup r)(x) = r(xg)$.

Lemma 3.5. $A_S(G, B) \simeq kG \otimes_{kS} B$ as G -algebras.

Proof. See [5, Proposition 3.3]. \square

Definition 3.6. Let us denote by $\text{Ind}_S^G(B)$ the G -algebra $A_S(G, B) \simeq kG \otimes_{kS} B$ and we will call the *induced algebra* from the S -algebra B .

Remark 3.7. Induction is a covariant functor: where to each homomorphism of S -algebras $f : A \rightarrow B$ is send to the homomorphism of G -algebras $\text{Ind}_S^G(f) : \text{Ind}_S^G(A) \rightarrow \text{Ind}_S^G(B)$, $\text{Ind}_S^G(f)(r) = f \circ r$.

Proposition 3.8. Let A be an imprimitive algebra, such that G acts transitively on the set $\{e_1, \dots, e_n\}$. If S is the stabilizer of e_1 then $A \simeq \text{Ind}_S^G(e_1 A)$ as G -algebras.

Proof. Let $\{g_1, \dots, g_n\}$ be a set of representatives for the right cosets G/S such that $g_i \rightharpoonup e_1 = e_i$. Then $\text{Ind}_S^G(e_1 A) = kG \otimes_{kS} e_1 A = \bigoplus_{i=1}^n g_i \otimes e_1 A$.

Consider the map

$$\begin{aligned} \psi : kG \otimes_{kS} e_1 A &\rightarrow A \\ \sum_{i=1}^n g_i \otimes e_1 a_i &\mapsto \sum_{i=1}^n e_i(g_i \rightharpoonup a_i). \end{aligned}$$

It is straightforward to check that the map ψ is a well defined G -algebra isomorphism. \square

Lemma 3.9. *Let S be a subgroup of G , and (B, \cdot) be an S -algebra. Then there is an algebra isomorphism between the induced algebra $\text{Ind}_S^G(B)$ and $\underbrace{B \times \cdots \times B}_{[G:S]}$. Moreover, $(\text{Ind}_S^G(B))^G \simeq B^S$.*

Proof. Let $n = [G : S]$ and $\{1_G, g_2, \dots, g_n\}$ be a set of representatives for the right cosets G/S . Considerer the functions $e_i \in \text{Ind}_S^G(B)$ defined by

$$(1) \quad e_i(x) = \begin{cases} 1_B, & \text{si } x \in Sg_i, \\ 0, & \text{si } x \notin Sg_i. \end{cases}$$

The set $\{e_1, \dots, e_n\}$ is an imprimitive system in $\text{Ind}_S^G(B)$, thus

$$\text{Ind}_S^G(B) \simeq e_1 \text{Ind}_S^G(B) \times \cdots \times e_n \text{Ind}_S^G(B).$$

It is clear that $B \simeq e_i \text{Ind}_S^G(B)$, so $\text{Ind}_S^G(B) \simeq \underbrace{B \times \cdots \times B}_{[G:S]}$.

Now, for the last claim, the map

$$\begin{aligned} B^S &\rightarrow (\text{Ind}_S^G(B))^G \\ b &\mapsto [x \mapsto b], \end{aligned}$$

is an algebra isomorphism. \square

Remark 3.10. We will say that the system $\{e_i\}$, defined in (1), is the *canonical imprimitive system associated to $\text{Ind}_S^G(B)$* .

3.2. Galois Objects as Induced Algebras. In this section we show that every Galois object of a finite group G is isomorphic to the induced algebra of a simple Galois object of a subgroup of G (see Theorem 3.14).

Lemma 3.11. *Let A be a semisimple G -algebra such that $A^G = k$. Then there is an imprimitive system $\{e_1, \dots, e_n\}$ in A , such that G acts transitively on this system, and $e_i A$ is a simple algebra, for all i .*

Proof. Applying Wedderburn's Theorem for semisimple algebras (see [8, Theorem 4, pag 157]) we conclude that there is a set $\{e_1, \dots, e_n\}$ of primitive orthogonal central idempotents in A such that $1_A = e_1 + \cdots + e_n$, and $e_i A$ is a simple algebra. It is immediate that $\{e_i\}$ is an imprimitive system in A . Suppose that G does not act transitivity on $\{e_i\}$. Renumbering if is necessary, we can suppose that the orbit of e_1 is $\{e_1, \dots, e_t\}$ and the orbit of e_{t+1} is $\{e_{t+1}, \dots, e_r\}$, where $1 \leq t < r \leq n$. Then $x_1 = e_1 + \cdots + e_t$ and $x_2 = e_{t+1} + \cdots + e_r$ are linearly independent elements in A^G . However, this contradicts the fact that $\dim_k A^G = 1$. \square

Remark 3.12. Following with the same hypothesis of the Lemma 3.11, and supposing $k = \mathbb{C}$, we have that $e_1 A = \text{End}_k(V)$ for some complex vector space V . Using the Skolem-Noether theorem, it is easy to see that every S -action over $\text{End}_k(V)$, is defined by a projective representation $\rho : S \rightarrow \text{PLG}(V)$,

$$(g \cdot f)(v) = \rho(g)(f(\rho(g)^{-1}(v))),$$

for $f \in \text{End}_k(V)$, $g \in G$, $v \in V$. Note that $\text{End}_k(V)^S = k$ if and only if V is an irreducible projective representation.

The following proposition is the Proposition 3.2 in [4], and we give an alternative proof.

Proposition 3.13. *Let S be a subgroup of G , and (B, \cdot) be an S -algebra. Then B is a k^S -Galois object if and only if the induced algebra $\text{Ind}_S^G(B)$ is a k^G -Galois object.*

Proof. We will use the Proposition 3.1 for the proof.

For the Lemma 3.9 and Proposition 3.1, it is clear that $B^S = k$ if and only if $\text{Ind}_S^G(B)^G = k$. Moreover, $\dim(B) = |S|$ if and only if $\dim(\text{Ind}_S^G(B)) = |G|$.

Let $\{e_1, \dots, e_n\}$ be the canonical imprimitive system in $\text{Ind}_S^G(B)$, see Remark 3.10.

Suppose that B is a k^S -Galois object. Let J be a G -invariant left ideal in $\text{Ind}_S^G(B)$. Then $e_1 J$ is an S -invariant left ideal of $e_1 \text{Ind}_S^G(B) \simeq B$. Since, $g_i \rightarrow (e_1 J) = e_i J$ then $J = \bigoplus_{i=1}^n e_i J$ is a trivial ideal.

Conversely, suppose that $\text{Ind}_S^G(B)$ is a k^G -Galois object. Let J be an S -invariant left ideal in $B \simeq e_1 \text{Ind}_S^G(B)$, since $\tilde{J} = g_1 J \oplus \dots \oplus g_n J$ is a G -invariant left ideal in $\text{Ind}_S^G(B)$. Then J is a trivial ideal. \square

Theorem 3.14. *Every k^G -Galois object is isomorphic as a G -algebra to the induced algebra $\text{Ind}_S^G(B)$, where B is a simple k^S -Galois object and S is a subgroup of G .*

Proof. Suppose that A is a k^G -Galois object. By Corollary 3.3 the algebra A is semisimple, so by Lemma 3.11 and Proposition 3.8 the k^G -Galois object A is the induced of a simple Galois object. Thus the theorem follows by Proposition 3.13. \square

Remark 3.15. Following the Remark 3.12, by Proposition 3.1 if V is an irreducible projective representation of the group S , the S -algebra $\text{End}_{\mathbb{C}}(V)$, is a \mathbb{C}^S -Galois object if and only $\dim_{\mathbb{C}}(V) = |S|^{1/2}$. The condition $\dim_{\mathbb{C}}(V) = |S|^{1/2}$ implies that the 2-cocycle of S , associated to the projective representation is non-degenerated, see Definition 4.4. Hence, in the complex case, the Galois objects of a group G are classified by pairs (S, α) , where $S \subseteq G$ is a subgroup, and $\alpha \in Z^2(S, \mathbb{C}^*)$ is a non-degenerated 2-cocycle, it is the main result of [10].

4. SIMPLE GALOIS OBJECTS

In this section we show a structure theorem for simple k^G -Galois objects. We use the Miyashita-Ulbrich action in order to prove that every simple Galois object is isomorphic as a graded algebra to a twisted group algebra of a normal subgroup of G .

4.1. Simple Galois objects as twisted group algebras. Recall that a k^G -module algebra B is the same as a G -graded algebra $B = \bigoplus_{g \in G} B_g$, where $B_g = B \triangleleft e_g$. Thus, if A is a k^G -Galois object, the Miyashita-Ulbrich action of k^G over A defines a structure of G -graded algebra $A = \bigoplus_{g \in G} A_g$, where

$$A_g = \{a \in A \mid ab = (g \rightharpoonup b)a \quad \forall b \in A\}.$$

Proposition 4.1. *Let A be a k^G -Galois object, then*

- i) $A_g A = A A_g$ is bilateral ideal of A ,
- ii) A_g is a $\mathcal{Z}(A)$ -submodule of A ,
- iii) $A_e = \mathcal{Z}(A)$,
- iv) $g \rightharpoonup A_x = A_{gxg^{-1}}$.

Proof. Straightforward. \square

From now on we will denote by A a k^G -Galois object, where A is a central simple algebra over K .

Let consider the normal subgroup of G , given by

$$(2) \quad N = \{x \in G \mid x \rightharpoonup \alpha = \alpha, \quad \forall \alpha \in K\}.$$

Lemma 4.2. *The following affirmations are equivalent:*

- i) $x \in N$.
- ii) $A_x \neq 0$.
- iii) $A_g A_x = A_{gx}$ for all $g \in G$.
- iv) $A_x A_{x^{-1}} = A_{x^{-1}} A_x = K$.
- v) There is an invertible element in A_x .

Proof. i) \Rightarrow ii) Since A is simple, by [13, Example 2.4] we have a category equivalence

$$A \otimes_K (-) : {}_K \mathcal{M} \rightarrow {}_A \mathcal{M}_A, \quad V \mapsto A \otimes V,$$

with quasi-inverse functor given by

$$(-)^A : {}_A \mathcal{M}_A \rightarrow {}_K \mathcal{M}, \quad M \mapsto M^A = \{m \in M \mid am = ma, \forall a \in A\}.$$

For every $x \in N$, we define an A -bimodule A^x , where the right action is the multiplication and left action is given by $b \cdot a = (x \rightharpoonup b)a$ for all $b \in A, a \in A^x$. Thus, $(A^x)^A = A_x = \{a \in A \mid ab = (x \rightharpoonup b)a, \quad \forall b \in A\}$, and since $A^x \neq 0$, then $A_x \neq 0$.

ii) \Rightarrow iii) Recall that $A_x A = A A_x$ is a bilateral ideal. Since A is simple, $0 \subsetneq A_x \subseteq A_x A = A$. The equation

$$\bigoplus_{g \in G} A_g = A = A_x A = \bigoplus_{g \in G} A_x A_g,$$

implies $A_x A_g = A_{xg}$.

iii) \Rightarrow iv) \Rightarrow v) Straightforward.

$v) \Rightarrow i)$ Let $u_x \in A_x$ be a unit, and let $\alpha \in K$. By the characterization of Miyashita-Ulbrich, we have

$$[(x \rightarrow \alpha) - \alpha]u_x = 0,$$

hence, $(x \rightarrow \alpha) - \alpha = 0$. \square

Remark 4.3. Let S be a finite group, K a field, and $\sigma \in Z^2(S, K^*)$ a 2-cocycle. For each $s \in S$, we will use the notation $u_s \in K_\sigma S$ to indicate the corresponding element in the *twisted group algebra* $K_\sigma S$. Thus $(u_s)_{s \in S}$ is a K -basis of $K_\sigma S$, and in this basis $u_s u_t = \sigma(s, t)u_{st}$.

Recall that an element $s \in S$ is called σ -regular if $\sigma(s, t) = \sigma(t, s)$ for all $t \in C_S(s)$. This definition depends only on the class of s under conjugation.

Definition 4.4. Let $\sigma \in Z^2(S, K^*)$ be a 2-cocycle. The 2-cocycle σ is called *non-degenerate* if and only if $\{1\}$ is the only σ -regular class in S .

Remark 4.5. The 2-cocycle σ is non-degenerate if and only if $\dim_K \mathcal{Z}(K_\sigma S) = 1$ (see [7, Theorem 9.3, pag 410]).

Proposition 4.6. *Let A be a k^G -Galois object, where A is a central simple algebra over K . Then A is isomorphic to a twisted group algebra $K_\sigma N$, where N is the normal subgroup of G that stabilizes K , and $\sigma : N \times N \rightarrow K^*$ is a non-degenerated 2-cocycle. Moreover, if we suppose $A = K_\sigma N$, then the action of $x \in N$ over $a \in A$ is given by*

$$x \rightarrow a = u_x a u_x^{-1}.$$

Proof. By the Lemma 4.2, A is a crossed ring over N . Since $A_e = \mathcal{Z}(A) = K$, A is isomorphic to a twisted group algebra $K_\sigma N$, thus by Remark 4.5 σ is a non-degenerate 2-cocycle.

Finally, if $x \in N$, then

$$y \in A_x = K u_x \leftrightarrow ya = (x \rightarrow a)y \quad \forall a \in A.$$

Taking $y = u_x$, we have $x \rightarrow a = u_x a u_x^{-1}$ for all $a \in A$. \square

4.2. The function γ . The group G acts over K through the natural projection,

$$(3) \quad G \rightarrow G/N \rightarrow \text{Gal}(K|k), \quad g \mapsto \bar{g}.$$

By item (iv) in Proposition 4.1, the action of G over A defines a function $\gamma : G \times N \rightarrow K^*$ determinate by the equation

$$(4) \quad g \rightarrow u_x = \gamma(g, x)u_{gx} \quad ({}^g x := gxg^{-1}).$$

Proposition 4.7. *For all $g, h \in G$ and $x, y \in N$, the function $\gamma : G \times N \rightarrow K^*$ defined in (4) satisfies the following equations:*

Condition C1:

$$(C1) \quad \gamma(x, y)\sigma(x, x^{-1}) = \sigma(x, y)\sigma(xy, x^{-1}).$$

Condition C2:

$$(C2) \quad \bar{g}(\sigma(x, y))\gamma(g, xy) = \sigma({}^g x, {}^g y)\gamma(g, x)\gamma(g, y).$$

Condition C3:

$$(C3) \quad \gamma(gh, x) = \bar{g}(\gamma(h, x))\gamma(g, {}^h x).$$

Proof. The conditions (C1), (C2), and (C3) are equivalent to

$$\begin{aligned} x \rightharpoonup u_y &= u_x u_y u_x^{-1}, \\ g \rightharpoonup u_x u_y &= (g \rightharpoonup u_x)(g \rightharpoonup u_y), \\ g \rightharpoonup (h \rightharpoonup u_x) &= gh \rightharpoonup u_x, \end{aligned}$$

respectively. \square

Remark 4.8. If $N \subseteq \mathcal{Z}(G)$, the condition (C1) is equivalent to

$$\gamma(x, y) = \frac{\sigma(x, y)}{\sigma(y, x)} =: \text{Alt}_\sigma(x, y).$$

Moreover, if $\sigma(N, N), \gamma(G, N) \subseteq k^*$, the conditions (C2) and (C3) are equivalent to

$$(5) \quad \gamma(g, xy) = \gamma(g, x)\gamma(g, y)$$

$$(6) \quad \gamma(gh, x) = \gamma(g, x)\gamma(h, x)$$

for all $g, h \in G$ and $x, y \in N$. A function γ that satisfies (5) and (6) is called a *pairing*.

5. CONSTRUCTION OF k^G -GALOIS OBJECTS FROM GROUP-THEORETICAL DATA

In this section we define group-theoretical data associated to a finite group G and a field k . We define a k^G -Galois object for each associate datum to G and k , and we establish an equivalence among the data, such that equivalent data define isomorphic Galois objects.

5.1. Data associated with Galois objects. Let G be a finite group, and let k be a field.

Definition 5.1. A *Galois datum associated to k^G* is a collection $(S, K, N, \sigma, \gamma)$ such that

- i) S is a subgroup of G and N is a normal subgroup of S .
- ii) $K \supseteq k$ is a Galois extension with Galois group S/N .
- iii) $\text{char}(k) \nmid |N|$.
- iv) $\sigma : N \times N \rightarrow K^*$ is a non-degenerate 2-cocycle.
- v) $\gamma : S \times N \rightarrow K^*$ satisfies the equations (C1), (C2), and (C3).

Let $(S, K, N, \sigma, \gamma)$ be a Galois datum associated to k^G . We will denote by $A(K_\sigma N, \gamma)$ the twisted group algebra $K_\sigma N$ with S -action defined by

$$g \rightharpoonup \alpha u_x = (g \rightharpoonup \alpha)(g \rightharpoonup u_x) = \bar{g}(\alpha)\gamma(g, x)u_{gx},$$

for $g \in S$, $x \in N$, and $\alpha \in K$.

We will denote by $\text{Ind}_S^G(A(K_\sigma N, \gamma))$ the induced G -algebra from the S -algebra $A(K_\sigma N, \gamma)$.

Proposition 5.2. *Let $(S, K, N, \sigma, \gamma)$ be a Galois datum associated to k^G . The S -algebra $A(K_\sigma N, \gamma)$ is a simple k^S -Galois object, with $\mathcal{Z}(A(K_\sigma N, \gamma)) = K$.*

Proof. Since $\text{char}(k) \nmid |N|$, by Maschke's Theorem for twisted group algebra (see [7, 2.10 Theorem]), the algebra $K_\sigma N$ is a semisimple algebra. Moreover, since σ is non-degenerate then $\mathcal{Z}(K_\sigma N) = K$. Hence $K_\sigma N$ is a central simple algebra over K .

Let us denote by A the S -algebra $A(K_\sigma N, \gamma)$. If we see that A is a K^N -Galois object, then A is a k^S -Galois object by Corollary 2.9.

Recall, if A is a central simple algebra over K , the map

$$\begin{aligned} A \otimes_K A^{op} &\rightarrow \text{End}_K(A) \\ a \otimes b &\mapsto [t \mapsto atb] \end{aligned}$$

is an isomorphism (see [6, pag. 32]). The map $\theta_N : A \# N \rightarrow \text{End}_K(A)$, $\theta_N(a \# x)(b) = a(x \rightharpoonup b)$ is surjective, since that

$$\theta_N(u_x u_y \# y^{-1})(a) = u_x a u_y.$$

Now, $\dim_K(A \# N) = \dim_K(\text{End}_K(A))$, hence θ_N is bijective. Thus, A is an K^N -Galois object by Proposition 2.3. \square

5.2. Equivalence of Galois data.

Definition 5.3. We will say that $(S, K, N, \sigma, \gamma)$ and $(S', K', N', \sigma', \gamma')$ are *equivalent Galois data associated to k^G* if there exists a G -algebra isomorphism between $\text{Ind}_S^G(A(K_\sigma N, \gamma))$ and $\text{Ind}_{S'}^G(A(K_{\sigma'} N', \gamma'))$.

Definition 5.4. Let G be a finite group, S a subgroup of G and (A, \cdot) an S -algebra. For each $g \in G$, we consider the $g^{-1}Sg$ -algebra $(A^{(g)}, \cdot_g)$, such that $A^{(g)} = A$ as algebras and $g^{-1}Sg$ -action give by

$$h \cdot_g a = (ghg^{-1}) \cdot a,$$

for all $h \in g^{-1}Sg$ and $a \in A^{(g)}$.

Lemma 5.5. *Let G be a finite group and S be a subgroup of G . For all $g \in G$, there exists a G -algebra isomorphism between $\text{Ind}_S^G(A)$ and $\text{Ind}_{g^{-1}Sg}^G(A^{(g)})$.*

Proof. For each $g \in G$, considerer the map

$$\begin{aligned}\psi_g : \text{Ind}_S^G(A) &\rightarrow \text{Ind}_{g^{-1}Sg}^G(A^{(g)}) \\ f &\mapsto \psi_g(f) = [h \mapsto f(gh)].\end{aligned}$$

It is easy to check that ψ_g is a G -algebra isomorphism. \square

Recall that two transitive G -sets G/S and G/S' are isomorphic if and only if there exists $g \in G$ such that $S' = g^{-1}Sg$.

Lemma 5.6. *Let S, S' be two subgroups of G , let A be a simple S -algebra, and B be a simple S' -algebra. Then there exists a G -algebra isomorphism between $\text{Ind}_S^G(A)$ and $\text{Ind}_{S'}^G(B)$ if and only if $S = g^{-1}S'g$ and $A \simeq B^{(g)}$ as S -algebras.*

Proof. Let $\{e_i\}_{1 \leq i \leq [G:S]}$, $\{r_i\}_{1 \leq i \leq [G:S']}$ and $\{e'_i\}_{1 \leq i \leq [G:g^{-1}S'g]}$ be the canonical imprimitive systems associated to $\text{Ind}_S^G(A)$, $\text{Ind}_{S'}^G(B)$ and $\text{Ind}_{g^{-1}S'g}^G(B^{(g)})$ (see Remark 3.10).

Suppose that $\chi : \text{Ind}_S^G(A) \rightarrow \text{Ind}_{S'}^G(B)$ is a G -algebra isomorphism. Since that A and B are simple algebras, the elements in $\{e_i\}_{1 \leq i \leq [G:S]}$, and $\{r_i\}_{1 \leq i \leq [G:S']}$ are the central primitive idempotents for $\text{Ind}_S^G(A)$, and $\text{Ind}_{S'}^G(B)$, respectively. Then χ induce a G -set isomorphism between $\{e_i\} \simeq G/S$ and $\{r_i\} \simeq G/S'$. Hence, there exist $g \in G$ such that $S = g^{-1}S'g$.

Now, by the Lemma 5.5 we have the following G -algebra isomorphism

$$\text{Ind}_S^G(A) \xrightarrow{\chi} \text{Ind}_{S'}^G(B) \xrightarrow{\psi_g} \text{Ind}_{g^{-1}S'g}^G(B^{(g)}),$$

then we have that

$$A \simeq e_1 \text{Ind}_S^G(A) \xrightarrow{\psi_g \circ \chi} e'_1 \text{Ind}_{S'}^G(B) \simeq B^{(g)}.$$

Conversely, suppose that $S = g^{-1}S'g$ and $A \simeq B^{(g)}$ as S -algebras. We can construct a G -algebra isomorphism between $\text{Ind}_S^G(A)$ and $\text{Ind}_{S'}^G(B)$, using the Remark 3.7 and the Lemma 5.5. \square

Let $f : A \rightarrow A'$ be an S -algebra isomorphism between $A = A(K_\sigma N, \gamma)$ and $A' = A(K'_\sigma N', \gamma')$. By the definition of the Miyashita-Ulbrich action, f is an S -graded algebra isomorphism. Since $N = \{x \in S \mid A_x \neq 0\}$ (see Lemma 4.2), then $N = N'$. Moreover, $f|_K$ is a field isomorphism, so we can suppose without loss of generality that $K = K'$.

Proposition 5.7. *The Galois data $(S, K, N, \sigma, \gamma)$ and $(S, K, N, \sigma', \gamma')$ associated to k^S are equivalent if and only if there are $\omega \in \mathcal{Z}(\text{Gal}(K|k))$, and $\eta : N \rightarrow K^*$, such that*

$$(7) \quad \omega(\sigma(x, y))\eta(xy) = \eta(x)\eta(y)\sigma'(x, y)$$

$$(8) \quad \omega(\gamma(s, x))\eta(sx) = \bar{s}(\eta(x))\gamma'(s, x),$$

for all $x, y \in N$, $s \in S$.

Proof. Let $\{u_x\}$ and $\{u'_x\}$ be a K -basis of A and A' , see Remark 4.3.

Suppose that $f : A(K_\sigma N, \gamma) \rightarrow A(K_{\sigma'} N, \gamma')$ be an S -algebra isomorphism. Considerer $\omega := f|_K \in \mathcal{Z}(\text{Gal}(K|k))$, and $\eta : N \rightarrow K^*$ determined by the equation

$$f(u_x) = \eta(x)u'_x \quad (\forall x \in N).$$

A straightforward computation shows that ω and η satisfy the equations (7), and (8).

Conversely, it is easy to check that if ω and η satisfy the equations (7), (8), then $f(\alpha u_x) := \omega(\alpha)\eta(x)u'_x$ is an S -algebra isomorphism. \square

Lemma 5.8. *Let $(S, K, N, \sigma, \gamma)$ be a Galois datum associated to k^G . For every $g \in G$, the Galois datum associated to $A(K_\sigma N, \gamma)^{(g)}$ is given by $(g^{-1}Sg, K, N, \sigma^{(g)}, \gamma^{(g)})$, where $\sigma^{(g)}(x, y) = \sigma(gxg^{-1}, gyg^{-1})$, and $\gamma^{(g)}(h, x) = \gamma(ghg^{-1}, gxg^{-1})$, for all $x, y \in N$, $h \in g^{-1}Sg$.*

Proof. Let $A = A(K_\sigma N, \gamma)$. Note that $(A^{(g)})_x = A_{gxg^{-1}}$, then $\{u_{gxg^{-1}} | x \in N\}$ is a basis of $A^{(g)}$, where $u_{gxg^{-1}} \in (A^{(g)})_x$. Hence, $\sigma^{(g)}(x, y) = \sigma(gxg^{-1}, gyg^{-1})$, and $\gamma^{(g)}(h, x) = \gamma(ghg^{-1}, gxg^{-1})$, for all $x, y \in N$, $h \in g^{-1}Sg$. \square

Theorem 5.9. *The data $(S, K, N, \sigma, \gamma)$ and $(S', K, N, \sigma', \gamma')$ are equivalent if and only if there exist $g \in G$, $\omega \in \mathcal{Z}(\text{Gal}(K|k))$ and $\eta : N \rightarrow K^*$ such that $S = g^{-1}S'g$ and*

$$\begin{aligned} \omega(\sigma(x, y))\eta(xy) &= \eta(x)\eta(y)\sigma'(gxg^{-1}, gyg^{-1}), \\ \omega(\gamma(s, x))\eta({}^s x) &= \bar{s}(\eta(x))\gamma'(gsg^{-1}, gxg^{-1}), \end{aligned}$$

for all $x, y \in N$, $s \in S$.

Proof. Since $A(K_\sigma N, \gamma)$ and $A(K_{\sigma'} N, \gamma')$ are simple algebras, the proof of theorem follows by Lemma 5.8, Lemma 5.6, and Proposition 5.7. \square

6. CLASSIFICATION OF k^G -GALOIS OBJECTS

In this section we aim to give a proof of our main result, *i.e.*, the Theorem 1.2.

Lemma 6.1. *Let $(S, K, N, \sigma, \gamma)$ be a Galois datum associated to k^S . If $A(K_\sigma N, \gamma)$ is a k^S -Galois object then $\text{char}(K) \nmid |N|$. In particular, $\text{char}(k) \nmid |N|$.*

Proof. By the Corollary 2.9, the algebra $K_\sigma N$ is a K^N -Galois object, so the map $\theta : K_\sigma N \rtimes N \rightarrow \text{End}(K_\sigma N)$ is an isomorphism. Note that

$$\theta\left(\sum_{y \in N} u_e \# y\right)(u_x) = \sum_{y \in N} u_y u_x u_y^{-1}$$

lies in the center of the twisted group algebra. Then $\theta(\sum_{y \in N} u_e \# y)(u_x) = 0$ if $x \neq e$ and $\theta(\sum_{y \in N} u_e \# y)(u_e) = |N|u_e$. Hence, $|N| \neq 0$. \square

Theorem 6.2. *Let S be a finite group, k be a field, and A be a simple S -algebra. Then A is a k^S -Galois object if and only if there exists $(S, K, N, \sigma, \gamma)$ a Galois datum associated to k^S , such that*

$$A \simeq A(K_\sigma N, \gamma),$$

as S -algebras.

Proof. Suppose that A is a k^S -Galois object. Let $K = \mathcal{Z}(A)$ and $N = \{g \in S | g \rightharpoonup \alpha = \alpha \ \forall \alpha \in K\}$. A is isomorphic to the twisted group algebra $K_\sigma N$, where σ is non-degenerate (see Proposition 4.6). Now, $\text{char}(k) \nmid |N|$ by Lemma 6.1. The function γ defined by the equation

$$g \rightharpoonup \alpha u_x = \bar{g}(\alpha) \gamma(g, x) u_{gx}, \quad (g \in S, x \in N, \alpha \in K),$$

completes a Galois datum $(S, K, N, \sigma, \gamma)$, such that $A \simeq A(K_\sigma N, \gamma)$. \square

Now our main result follows immediately from our previous results.

Proof of Theorem 1.2. It follows from Theorem 6.2, and Theorem 3.14. \square

6.1. Two families of Examples. Let $G = \mathbb{Z}_n \oplus \mathbb{Z}_n \oplus \mathbb{Z}_n$ and $N = \{0\} \oplus \mathbb{Z}_n \oplus \mathbb{Z}_n \subseteq G$ with n an odd prime. Considerer the fields $k = \mathbb{Q}[\zeta]$ and $K = \mathbb{Q}[q]$, where $q = e^{\frac{2\pi i}{n^2}}$ is a primitive n th root of unity and $\zeta = q^n$. Now, define a 2-cocycle $\sigma : N \times N \rightarrow k^*$ of N with values in $\langle \zeta \rangle^* \subseteq k^*$, by

$$\sigma(x, y) = \zeta^{x_2 y_1 - x_1 y_2} \quad (x = (x_1, x_2), y = (y_1, y_2) \in N).$$

Since $\{0\} = \{x \in N | \sigma(x, y) = \sigma(y, x) \ \forall y \in N\}$, thus σ is a non-degenerate 2-cocycle.

In this case, by the Remark 4.8 a function $\gamma : G \times N \rightarrow k^* \subseteq K^*$ satisfies the conditions (C1), (C2) and (C3) if and only if γ is a pairing such that $\gamma|_{N \times N} = \text{Alt}_\sigma = \sigma^2$. Hence, it is easy to see that the collection $(G, K, N, \sigma, \gamma)$ is a Galois datum associated to k^G . Moreover, if γ' is another paring such that $\gamma'|_{N \times N} = \sigma^2$, then the Galois data $(G, K, N, \sigma, \gamma)$ and $(G, K, N, \sigma, \gamma')$ are equivalent if and only if $\gamma = \gamma'$.

We can obtain other family of Galois data taking G , N , k , and K as above, and defining the 2-cocycle $\sigma' : N \times N \rightarrow k^*$ by

$$\sigma'(x, y) = \zeta^{x_2 y_1} \quad (x = (x_1, x_2), y = (y_1, y_2) \in N),$$

and a pairing $\gamma : G \times N \rightarrow k^*$, such that

$$\gamma(x, y) = \text{Alt}_{\sigma'}(x, y) = \zeta^{x_2 y_1 - x_1 y_2},$$

for all $x, y \in N$.

7. OBSTRUCTION THEORY FOR GALOIS DATA

The main result of this section is to present some necessities conditions for the existence of a function $\gamma : G \times N \rightarrow K^*$, such that $(S, N, K, \sigma, \gamma)$ is a Galois datum associated to k^G .

7.1. Hochschild cohomology for groups. We briefly recall the well-known Hochschild cohomology for the algebra $\mathbb{Z}G$.

Let G be a group and A be a G -bimodule. Define $C^0(G, A) = A$, and for $n \geq 1$

$$C^n(G, A) = \{f : \underbrace{G \times \cdots \times G}_{n\text{-times}} \rightarrow A \mid f(x_1, \dots, x_n) = 0, \text{ if } x_i = 1_G \text{ for some } i\}.$$

Considerer the following cochain complex

$$0 \longrightarrow C^0(G, A) \xrightarrow{d_0} C^1(G, A) \xrightarrow{d_1} C^2(G, A) \cdots C^n(G, A) \xrightarrow{d_n} C^{n+1}(G, A) \cdots$$

where

$$\begin{aligned} d_n(f)(x_1, x_2, \dots, x_{n+1}) &= x_1 \rightharpoonup f(x_2, \dots, x_{n+1}) \\ &\quad + \sum_{i=1}^n (-1)^i f(x_1, \dots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \dots, x_{n+1}) \\ &\quad + (-1)^{n+1} f(x_1, \dots, x_n) \leftarrow x_{n+1}. \end{aligned}$$

Then as usual define $HZ^n(G, A) := \ker(d_n)$, $HB^n(G, A) := \text{Im}(d_{n-1})$ and $HH^n(G, A) := HZ^n(G, A)/HB^n(G, A)$ ($n \geq 0$) the Hochschild cohomology of G with coefficients in A .

Remark 7.1. A left G -module A , is a G -bimodule with the trivial right action, in this case the differential maps will be denoted by δ_n , and the Hochschild Cohomology of G with coefficients in A , is the ordinary group cohomology.

Let G be a finite group, N a normal subgroup of G , K a Galois extension of k with Galois group G/N , and $\sigma \in Z^2(N, K^*)$ a non-degenerate 2-cocycle.

Thus the abelian group

$$C^1(N, K^*) = \{f : N \rightarrow K^* \mid f(e) = 1\},$$

is a G -bimodule with left action

$$(9) \quad (g \rightharpoonup f)(x) = \bar{g}(f(x)),$$

and right action

$$(10) \quad (f \leftarrow g)(x) = f({}^g x) \quad ({}^g x = gxg^{-1}),$$

for all $g \in G, x \in N$, and $f \in C^1(N, K^*)$.

By abuse of notation we will identify to a function $\gamma : G \rightarrow C^1(N, K^*)$ with its associated function $\gamma : G \times N \rightarrow K^*$, and vice versa.

7.2. Obstructions. The abelian groups $C^n(N, K^*)$, $Z^n(G, K^*)$ are G -bimodules with left action $(g \cdot \sigma)(x_1, \dots, x_n) = \bar{g}(\sigma(x_1, \dots, x_n))$, and right action $(\sigma^g)(x_1, \dots, x_n) = \sigma({}^g x_1, \dots, {}^g x_n)$. Analogously, the abelian group $\text{Hom}(N, K^*) = \hat{N}$ is a G -bimodule.

Proposition 7.2 (First obstruction). *There exists a function $\gamma : G \times N \rightarrow K^*$ that satisfied (C2) if and only if the second cohomology class of $\frac{g \cdot \sigma}{\sigma^g}$ is zero for all $g \in G$.*

Proof. It follows immediately by the condition (C2). \square

Suppose that the first obstruction is zero, then for all $g \in G$ there exists $\gamma_g : N \rightarrow K^*$, such that

$$\delta_1(\gamma_g) = \frac{g \cdot \sigma}{\sigma^g} \in B^2(N, K^*),$$

it defines a function $\gamma : G \rightarrow C^1(N, K^*)$, $g \mapsto \gamma_g$.

Lemma 7.3. *For all $g, h \in G$, $d_1(\gamma)(g, h) \in \widehat{N}$, the function $d_1(\gamma) : G \times G \rightarrow \widehat{N}$ is a Hochschild 2-cocycle of G with values in \widehat{N} .*

Proof. If $g, h \in G$, then

$$\begin{aligned} \delta_1(d_1(\gamma)(g, h)) &= \delta_1[(g \cdot \gamma_h)(\gamma_{gh})^{-1}(\gamma_g^h)] \\ &= \left(g \cdot \frac{h \cdot \sigma}{\sigma^h}\right) \left(\frac{\sigma^{gh}}{gh \cdot \sigma}\right) \left(\frac{g \cdot \sigma^h}{(\sigma^g)^h}\right) \\ &= 1. \end{aligned}$$

\square

A straightforward computation shows that if $\gamma' : G \rightarrow C^1(N, K^*)$, $g \mapsto \gamma'_g$, is another function such that $\delta_1(\gamma'_g) = \frac{g \cdot \sigma}{\sigma^g}$ for all $g \in G$, then the Hochschild 2-cocycles $d_1(\gamma')$ and $d_1(\gamma)$ are cohomologous.

Proposition 7.4 (Second obstruction). *There exists a function $\gamma : G \times N \rightarrow K^*$ that satisfied (C3) if and only if the second Hochschild cohomology class of $d_1(\gamma)$ is zero.*

Proof. If there is a function $\gamma : G \times N \rightarrow K^*$, that satisfies (C3), then $d_1(\gamma) = 0$, so its cohomology is zero too.

Conversely, if $\gamma \in C^1(G, C^1(N, K^*))$ is a function, such that there exists $\theta \in C^1(G, \widehat{N})$ with $d_1(\theta) = d_1(\gamma)$, then the function $\bar{\gamma} : G \times N \rightarrow K^*$ defined by $\bar{\gamma}(g, n) = \theta_g(n)^{-1} \gamma(g, n)$ for all $g \in G, n \in N$, satisfies (C3). \square

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